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Diagonal dominance of damping and the decoupling approximation in linear vibratory systems

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Abstract

A common approximation in the analysis of non-classically damped systems is to ignore the off-diagonal elements of the modal damping matrix. This procedure is termed the decoupling approximation. It is generally believed that errors due to the decoupling approximation should be negligible if the modal damping matrix is diagonally dominant. In addition, the errors are expected to decrease as the modal damping matrix becomes more diagonally dominant. It is shown numerically in this paper that, over a finite range, errors due to the decoupling approximation can increase monotonically at any specified rate while the modal damping matrix becomes more diagonally dominant with its off-diagonal elements decreasing continuously in magnitude. An explanation for these unexpected drifts of decoupling errors is provided with the use of complex coupling coefficients. Small off-diagonal elements in the modal damping matrix are not sufficient to ensure small errors due to the decoupling approximation. Any error-criterion based solely upon the diagonal dominance of the modal damping matrix would not be accurate.

1. Introduction

It is well known that an undamped linear system possesses classical normal modes, and that in each mode different parts of the system vibrate with the same frequency, passing through their equilibrium positions at the same time. The normal modes constitute a modal matrix, which defines a linear coordinate transformation that decouples the undamped system. This process of decoupling the equation of motion of an undamped vibratory system is a time-honored procedure termed modal analysis. Upon decoupling, an undamped linear system can be treated as a series of independent single-degree-of-freedom systems.

In the presence of damping, a linear system cannot be decoupled by modal analysis unless it possesses a full set of classical normal modes, in which case the system is said to be classically damped [1]. Practically speaking, classical damping means that energy dissipation is almost uniformly distributed throughout the system. This assumption is violated for systems consisting of two or more parts with significantly different

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levels of damping. Examples of such systems include soil-structure systems [2], base-isolated structures [3,4], and systems in which coupled vibrations of structures and fluids occur. Increasing use of special energy-dissipating devices in control constitutes another important example. In fact, experimental modal testing suggests that no real physical system is strictly classically damped [5]. Thus damped linear systems cannot in general be decoupled by the classical modal transformation. In addition, it has been showed [6] that no time-invariant linear transformations in the configuration space can decouple all damped systems. Even partial decoupling, i.e. simultaneous transformation of the coefficient matrices of the equation of motion to upper triangular forms, cannot be ensured with time-invariant linear transformations in the configuration space [7].

Classical normal modes are all real. Thus modal analysis in the classical sense involves a real transformation. Foss and others [8–10] extended classical modal analysis to a process of complex modal analysis in the state space to treat non-classically damped systems. However, the state-space approach has not appealed to practicing engineers. There are several reasons for this situation. First, a common excuse is that the state-space approach is computationally more involved because the dimension of the state space is twice the number of degrees of freedom. Second, complex modal analysis still cannot decouple all non-classically damped systems. A condition of non-defective eigenvectors in the state space must be satisfied in order for complex modal analysis to achieve complete decoupling in the state space. Third, and perhaps more importantly, there is little physical insight associated with different elements of complex modal analysis. Classical modal analysis is amenable to physical interpretation. For example, each classical normal mode represents a physical profile of vibration. Even the eigenvalue problem associated with classical modal analysis can be interpreted geometrically as a problem of finding the principal axes of an ellipsoid.

The continuing popularity of modal analysis leads to a predictable observation: engineers have devised a whole array of ingenious but rather bold approximations [11–18] to base the analysis of linear systems upon the classical modal transformation. Among the various approximations, the simplest one may be attributed to Lord Rayleigh. In Section 102 of "The Theory of Sound" in 1894, Rayleigh [19] suggested that the off-diagonal elements of the modal damping matrix may be neglected if they are small in magnitude. This simple procedure, termed the decoupling approximation [18,20,21], is a member of the class of approximate techniques seeking to decouple a damped system by modifying its coefficient matrices. Knowles [22] showed that the decoupling approximation is optimal in the sense that the difference (matrix norm) between the original equation and the equation obtained by the decoupling approximation is minimized. It does not mean, however, that errors due to the decoupling approximation are minimized. Contrary to intuition and to widely accepted beliefs, it will be shown in this paper that errors due to the decoupling approximation can increase monotonically at any specified rate while the modal damping matrix becomes more diagonal dominant. The organization of the paper is as follows. In Section 2, a concise review of the decoupling approximation is provided. Diagonal dominance of modal damping matrices is discussed in Section 3, in which two indices are introduced to quantify the degree of being diagonal. Examples are constructed in Section 4 to demonstrate that errors due to the decoupling approximation can drift around in an unexpected fashion. An explanation for the drifts in errors is offered in Section 5. In Section 6, a summary of major findings is given.

2. Coordinate coupling and the decoupling approximation

The equation of motion of an n-degree-of-freedom linear system can be written in the form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t) \tag{1}$$

with initial conditions $\mathbf{x}(0) = \mathbf{x}_0$, $\dot{\mathbf{x}}(0) = \dot{\mathbf{x}}_0$. The generalized coordinate $\mathbf{x}$ and excitation $\mathbf{f}(t)$ are real n-dimensional column vectors. The mass matrix $\mathbf{M}$, the damping matrix $\mathbf{C}$, and the stiffness matrix $\mathbf{K}$ are real matrices of order $n \times n$. For passive systems, $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are symmetric and positive definite. Associated with the undamped system is a generalized eigenvalue problem [18]

$$\mathbf{K}\mathbf{u} = \lambda\mathbf{M}\mathbf{u}. \tag{2}$$

Owing to the definiteness of the coefficient matrices, all eigenvalues λ_i are real and positive, and the corresponding eigenvectors $\mathbf{u}_j$ are real and orthogonal with respect to either $\mathbf{M}$ or $\mathbf{K}$. The n eigenvectors $\mathbf{u}_j$

constitute the modal matrix

$$\mathbf{U} = [\mathbf{u}_1|\mathbf{u}_2|\cdots|\mathbf{u}_n]. \tag{3}$$

The spectral matrix of system (1) is defined as a diagonal matrix

$$\mathbf{\Omega} = \mathrm{diag}[\omega_1^2, \omega_2^2, \ldots, \omega_n^2] \tag{4}$$

consisting of the eigenfrequencies squared such that $\omega_i^{\,2} = \lambda_i$. Upon normalization, the orthogonality of the modes can be expressed in a compact form:

$$\mathbf{U}^{\mathrm{T}}\mathbf{M}\mathbf{U} = \mathbf{I}, \tag{5}$$

$$\mathbf{U}^{\mathrm{T}}\mathbf{K}\mathbf{U} = \mathbf{\Omega}. \tag{6}$$

The modal matrix $\mathbf{U}$ defines a real invertible coordinate transformation

$$\mathbf{x} = \mathbf{U}\mathbf{q}. \tag{7}$$

In terms of the principal coordinate $\mathbf{q}$, the equation of motion of a damped system takes the canonical form

$$\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{\Omega}\mathbf{q} = \mathbf{g}(t) \tag{8}$$

with initial conditions $\mathbf{q}(0) = \mathbf{U}^{\mathrm{T}}\mathbf{M}\mathbf{x}_0$, $\dot{\mathbf{q}}(0) = \mathbf{U}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{x}}_0$ and excitation $\mathbf{g}(t) = \mathbf{U}^{\mathrm{T}}\mathbf{f}(t)$. The symmetric matrix

$$\mathbf{D} = \mathbf{U}^{\mathrm{T}}\mathbf{C}\mathbf{U} \tag{9}$$

is referred to as the modal damping matrix. While an undamped system can be decoupled entirely by modal analysis, a damped system is completely decoupled if and only if $\mathbf{D}$ is diagonal. Any coupling in a linear system occurs ultimately through damping.

A system is classically damped if it can be decoupled by classical modal analysis. Rayleigh [19] asserted that a system is classically damped if $\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$ for some scalar constants α and β. This requirement, referred to as proportional damping, is sufficient but not necessary for classical damping. In 1965, Caughey and O'Kelly [1] established that a necessary and sufficient condition under which a system is classically damped is

$$\mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{C}. \tag{10}$$

There is, however, no reason why the above condition should be satisfied. In general, a linear vibratory system is non-classically damped and the modal damping matrix $\mathbf{D}$ cannot be diagonalized by classical modal analysis.

Express the modal damping matrix in the form

$$\mathbf{D} = \mathbf{D}_d + \mathbf{D}_o, \tag{11}$$

where $\mathbf{D}_d = \mathrm{diag}[d_{11}, d_{22}, \ldots, d_{nn},]$ is a diagonal matrix composed of the diagonal elements of $\mathbf{D}$, and $\mathbf{D}_o$ is a matrix with zero diagonal elements and whose off-diagonal elements coincide with those in $\mathbf{D}$. The decoupling approximation, as expounded by Rayleigh [19], amounts to simply neglecting $\mathbf{D}_o$ and thus replacing $\mathbf{D}$ by $\mathbf{D}_d$. The system response by the decoupling approximation satisfies the decoupled equation

$$\ddot{\mathbf{q}}_a(t) + \mathbf{D}_d\dot{\mathbf{q}}_a(t) + \mathbf{\Omega}\mathbf{q}_a(t) = \mathbf{g}(t) \tag{12}$$

with initial conditions $\mathbf{q}_a(0) = \mathbf{q}(0)$, $\dot{\mathbf{q}}_a(0) = \dot{\mathbf{q}}(0)$. The error due to the decoupling approximation is defined as the difference of the exact and approximate solutions:

$$\mathbf{e}(t) = \mathbf{q}(t) - \mathbf{q}_a(t). \tag{13}$$

Subtract Eq. (12) from Eq. (8) to obtain

$$\ddot{\mathbf{e}}(t) + \mathbf{D}_d\dot{\mathbf{e}}(t) + \mathbf{\Omega}\mathbf{e}(t) = -\mathbf{D}_o\dot{\mathbf{q}}(t) \tag{14}$$

with initial conditions $\mathbf{e}(0) = 0$, $\dot{\mathbf{e}}(0) = 0$. The decoupling approximation is just a member of the class of approximate techniques seeking to decouple a damped system by modifying its coefficient matrices [23–27]. It is optimal in the sense that the difference (matrix norm) between Eqs. (8) and (12) is minimized [22]. This does not imply, however, that errors due to the decoupling approximation are minimized. Owing to its simplicity, the decoupling approximation enjoys popular use in structural dynamics.

Because of its practical importance, many authors studied the implications of the decoupling approximation and several indices were developed to quantify the extent of modal coupling in non-classically damped systems. These coupling indices are sometimes referred to as non-proportionality indices or coupling indices. For example, various indices were proposed by Prater and Singh [28], Nair and Singh [29], Tong et al. [30], Bellos and Inman [31] and Bhaskar [32]. Using the frequency-domain approach, Hasselman [33] established a criterion for determining whether a non-classically damped system may be regarded as practically decoupled. A similar criterion was suggested by Warburton and Soni [34]. It was also shown [35,36] that for greatly separated frequencies and small damping, the Euclidean norm of the errors due to the decoupling approximation in each mode is small. The influence of excitation frequencies on errors due to the decoupling approximation was examined by Park et al. [37]. It is suggested that frequency avoidance, rather than frequency separation, is an efficient method to control the decoupling errors. Coupling Indices based solely upon the complex modes of non-classically damped systems were derived by Liu et al. [38,39], Prells and Friswell [40], Adhikari [41], and Bhaskar [42]. Coupling indices may sometimes be used to predict the errors due to the decoupling approximation.

In order to evaluate the errors due to the decoupling approximation, apply Fourier transform to Eq. (14) to get

$$\mathbf{E}(\mathrm{i}\omega) = -\mathrm{i}\omega\mathbf{H}_a(\mathrm{i}\omega)\mathbf{D}_o\mathbf{Q}(\mathrm{i}\omega), \tag{15}$$

where

$$\mathbf{H}_a(\mathrm{i}\omega) = (\mathbf{\Omega} - \omega^2\mathbf{I} + \mathrm{i}\omega\mathbf{D}_d)^{-1} \tag{16}$$

is the frequency response matrix of the decoupled system (12), and $\mathbf{Q}(\mathrm{i}\omega)$ is the Fourier transform of the modal coordinate $\mathbf{q}(t)$. As a standard notation, Fourier transforms are denoted by capital letters, $\mathbf{I}$ represents the $n \times n$ identity matrix and $\mathrm{i} = \sqrt{-1}$. Using any vector norm, $\|\mathbf{E}(\mathrm{i}\omega)\|$ quantifies the effects of modal coupling in the system. On the other hand, the magnitude of the kth element in the complex vector $\mathbf{E}(\mathrm{i}\omega) = [E_1(\mathrm{i}\omega), \ldots, E_n(\mathrm{i}\omega)]^{\mathrm{T}}$ is a measure of the effects of modal coupling in that mode. Upon normalization with respect to $\|\mathbf{Q}(\mathrm{i}\omega)\|$, the quantity

$$s(\mathrm{i}\omega) = \frac{\|\mathbf{E}(\mathrm{i}\omega)\|}{\|\mathbf{Q}(\mathrm{i}\omega)\|} = \frac{\|\omega\mathbf{H}_a(\mathrm{i}\omega)\mathbf{D}_o\mathbf{Q}(\mathrm{i}\omega)\|}{\|\mathbf{Q}(\mathrm{i}\omega)\|} \tag{17}$$

may be interpreted as the relative steady-state error due to the decoupling approximation. In subsequent analysis, the Euclidean norm is chosen for convenience. However, any other norm will yield the same qualitative results.

3. Quantification of diagonal dominance

It is generally accepted that errors due to the decoupling approximation must be small if the off-diagonal elements of the modal damping matrix $\mathbf{D}$ are small [18–21]. In addition, the errors are expected to decrease as $\mathbf{D}$ becomes more diagonally dominant. But the meanings of these terms are not clear. When does a matrix become more diagonally dominant than another? How can one quantify the property of being diagonal? These issues will first be clarified.

The modal damping matrix $\mathbf{D}$ is said to be diagonally dominant [43] if

$$|d_{ii}| \geqslant \sum_{\substack{j=1 \\ j \neq i}}^{n} |d_{ij}| \tag{18}$$

for $1 \leqslant i \leqslant n$. It is said to be strictly diagonally dominant if strict inequality holds. These definitions of diagonal dominance have solid footing in linear algebra and many important properties of diagonally dominant matrices have been established. For example, if $\mathbf{D}$ is diagonal dominant, then the real parts of its eigenvalues have the same sign as the diagonal entries. The concept of diagonal dominance was generalized [44] in recent years. The matrix $\mathbf{D}$ is diagonally dominant in a generalized sense if there exist non-zero scalars α_i

such that

$$|d_{ii}| \geqslant \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\alpha_i}{\alpha_j} |d_{ij}|, \quad |d_{ii}| \geqslant \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\alpha_j}{\alpha_i} |d_{ij}|, \tag{19}$$

for $1 \leqslant i \leqslant n$. Clearly, a diagonally dominant matrix is diagonally dominant in the generalized sense. Recall the definitions of $\mathbf{D}_d$ and $\mathbf{D}_o$ in Eq. (11). Let $|\mathbf{D}_d|$ and $|\mathbf{D}_o|$ be matrices whose elements are the absolute values of those in $\mathbf{D}_d$ and $\mathbf{D}_o$, respectively. It can be shown [45] that if

$$\sigma\left(\left|\mathbf{D}_d^{-1}\right||\mathbf{D}_o|\right) < 1, \tag{20}$$

where $\sigma\left(\left|\mathbf{D}_d^{-1}\right||\mathbf{D}_o|\right)$ is the spectral radius (largest absolute value of any eigenvalue) of $\left|\mathbf{D}_d^{-1}\right||\mathbf{D}_o|$, then $\mathbf{D}$ is diagonally dominant in the generalized sense. The concept of generalized diagonal dominance has broad applications in multivariable control theory.

Although the concepts of diagonal dominance are widely used, numerical indices for quantifying the degree of being diagonal have not been reported in the open literature. Based upon Eq. (18), an index of diagonality may be readily defined as

$$\rho(\mathbf{D}) = \frac{\sum_{i=1}^{n} |d_{ii}|}{\sum_{\substack{i,j=1 \\ i\neq j}}^{n} |d_{ij}|}. \tag{21}$$

Clearly, $0 \leqslant \rho(\mathbf{D}) \leqslant \infty$. If a modal damping matrix $\mathbf{D}$ is diagonally dominant then $\rho(\mathbf{D}) \geqslant 1$ and, for a diagonal $\mathbf{D}$, $\rho(\mathbf{D}) = \infty$. A second index of diagonality, based upon Eq. (20), is defined as

$$\rho_1(\mathbf{D}) = \sigma\left(\left|\mathbf{D}_d^{-1}\right||\mathbf{D}_o|\right). \tag{22}$$

Note that the two indices $\rho(\mathbf{D})$ and $\rho_1(\mathbf{D})$ have opposite trends: $\mathbf{D}$ is diagonally dominant in the generalized sense if and only if $0 \leqslant \rho_1(\mathbf{D}) \leqslant 1$ and, for a diagonal $\mathbf{D}$, $\rho_1(\mathbf{D}) = 0$. If $|\mathbf{D}|$ is irreducible, i.e. $d_{ij} \neq 0$ for $i \neq j$, then $\rho_1(\mathbf{D})$ is monotonic increasing as the off-diagonal elements of $\mathbf{D}$ increase in magnitude [45]. In addition,

$$\min_i \sum_{\substack{j=1 \\ j\neq i}}^{n} \left|\frac{d_{ij}}{d_{ii}}\right| \leqslant \rho_1(\mathbf{D}) \leqslant \max_i \sum_{\substack{j=1 \\ j\neq i}}^{n} \left|\frac{d_{ij}}{d_{ii}}\right|. \tag{23}$$

An important advantage of $\rho_1(\mathbf{D})$ is that it lies within a finite range for diagonally dominant matrices. On the other hand, $\rho(\mathbf{D})$ can be computed more easily.

It is possible to define other indices of diagonality. However, it will become evident that the choice of an index of diagonality for $\mathbf{D}$ is immaterial in characterizing the drifts of errors due to the decoupling approximation. It must be kept in mind that neither $\rho(\mathbf{D})$ nor $\rho_1(\mathbf{D})$ are intended for measuring decoupling errors, only how diagonal the modal damping matrix $\mathbf{D}$ is.

4. Drifts in errors due to the decoupling approximation

It will now be demonstrated numerically that, contrary to intuition, errors due to the decoupling approximation can increase monotonically at any specified rate while the modal damping matrix $\mathbf{D}$ becomes more diagonally dominant with its off-diagonal elements decreasing continuously in magnitude.

Example 1. Consider two four-degree-of-freedom systems of the form (8). System 1 is governed by [46]

$$\ddot{\mathbf{q}} + \mathbf{D}_1\dot{\mathbf{q}} + \mathbf{\Omega}_1\mathbf{q} = \mathbf{g}(t), \tag{24}$$

where the spectral matrix, the modal damping matrix, and the excitation are given by

$$\mathbf{\Omega}_1 = \mathrm{diag}[3.95^2, 3.98^2, 4.00^2, 4.10^2], \tag{25}$$

$$\mathbf{D}_1 = \begin{bmatrix} 1.61 & -0.1865 & -0.1742 & 0.3838 \\ -0.1865 & 1.7 & 0.3792 & -0.1773 \\ -0.1742 & 0.3792 & 1.8 & -0.1742 \\ 0.3838 & -0.1773 & -0.1742 & 1.75 \end{bmatrix}, \tag{26}$$

$$\mathbf{g}(t) = \hat{\mathbf{g}}\cos(\omega t) = [1 \quad 1 \quad 1 \quad 1]^{\mathrm{T}} \cos(4.16t). \tag{27}$$

System 2 is governed by

$$\ddot{\mathbf{q}} + \mathbf{D}_2\dot{\mathbf{q}} + \mathbf{\Omega}_1\mathbf{q} = \mathbf{g}(t) \tag{28}$$

which differs from Eq. (24) only in the modal damping matrix

$$\mathbf{D}_2 = \begin{bmatrix} 1.61 & 0.0009 & 0.04 & 0.041 \\ 0.0009 & 1.7 & 0.0227 & 0.0376 \\ 0.04 & 0.0227 & 1.8 & 0.04 \\ 0.041 & 0.0376 & 0.04 & 1.75 \end{bmatrix}. \tag{29}$$

While both $\mathbf{D}_1$ and $\mathbf{D}_2$ satisfy Eq. (18) and are diagonally dominant, $\mathbf{D}_2$ is a lot more diagonal. This is perhaps obvious since $\mathbf{D}_1$, $\mathbf{D}_2$ have the same diagonal elements but each off-diagonal element in $\mathbf{D}_2$ is at least an-order-of-magnitude smaller than the corresponding element in $\mathbf{D}_1$. Utilizing the indices of diagonality, it is found that

$$\rho(\mathbf{D}_1) = 2.32 \ll 18.83 = \rho(\mathbf{D}_2), \tag{30}$$

$$\rho_1(\mathbf{D}_1) = 0.43 \gg 0.055 = \rho_1(\mathbf{D}_2). \tag{31}$$

Remember that $\rho(\mathbf{D})$ and $\rho_1(\mathbf{D})$ have opposite trends; both indicate that $\mathbf{D}_2$ is significantly more diagonally dominant than $\mathbf{D}_1$. Intuitively, one would expect System 2 to yield a smaller error due to the decoupling approximation than System 1. However, calculation of the steady-state errors due to the decoupling approximation yields an opposite result:

$$s_1(\mathrm{i}\omega) = 2.76\% < 5.31\% = s_2(\mathrm{i}\omega). \tag{32}$$

Hence, errors due to the decoupling approximation can be larger for systems whose modal damping matrix is more diagonal.

4.1. Linear interpolation between damping matrices

This example can be extended. Define a series of systems

$$\ddot{\mathbf{q}} + \mathbf{D}_\alpha\dot{\mathbf{q}} + \mathbf{\Omega}_1\mathbf{q} = \mathbf{g}(t) \tag{33}$$

indexed by a parameter α in such a way that

$$\mathbf{D}_\alpha = (1-\alpha)\mathbf{D}_1 + \alpha\mathbf{D}_2, \quad 0 \leqslant \alpha \leqslant 1. \tag{34}$$

As α increases from 0 to 1, $\mathbf{D}_\alpha$ is linearly interpolated between $\mathbf{D}_1$ and $\mathbf{D}_2$ with the diagonal entries of $\mathbf{D}_\alpha$ remaining fixed. For $0 \leqslant \alpha \leqslant 1$, the steady-state error $s_\alpha(\mathrm{i}\omega)$ due to the decoupling approximation is computed. In Fig. 1, the decoupling error $s_\alpha(\mathrm{i}\omega)$ is plotted against the index of diagonality $\rho(\mathbf{D}_\alpha)$. It is observed that as the modal damping matrix becomes more diagonally dominant, the error due to the decoupling approximation increases monotonically. To be specific, the error $s_\alpha(\mathrm{i}\omega)$ due to the decoupling approximation increases continuously from 2.76% to 5.31% as the index of diagonality $\rho(\mathbf{D}_\alpha)$ increases continuously from 2.32 to 18.83.

If the choice of an index of diagonality is immaterial, one should be able to obtain qualitatively identical results using the second index of diagonality $\rho_1(\mathbf{D}_\alpha)$. Since $\rho_1(\mathbf{D}_\alpha)$ and $\rho(\mathbf{D}_\alpha)$ have opposite trends, the decoupling error $s_\alpha(\mathrm{i}\omega)$ is plotted against $1/\rho_1(\mathbf{D}_\alpha)$ in Fig. 2. It is observed that the error $s_\alpha(\mathrm{i}\omega)$ increases

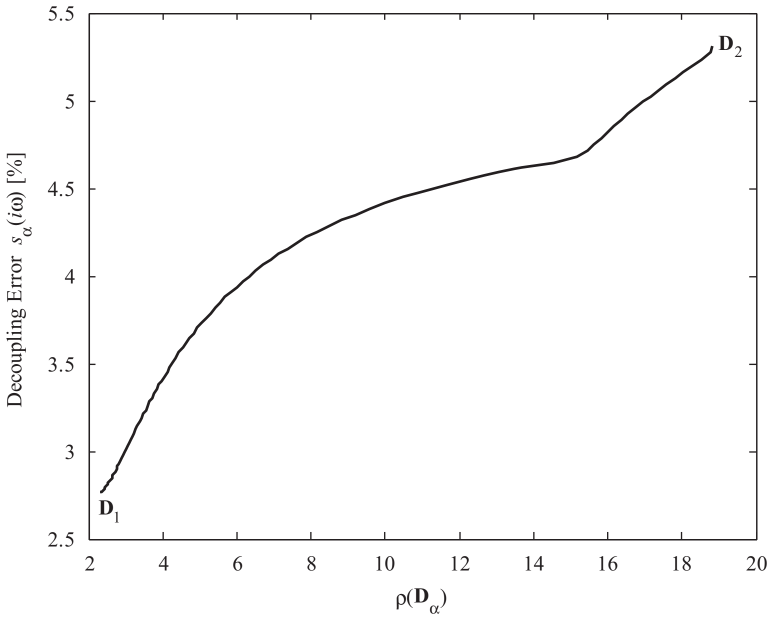

Fig. 1. Error due to the decoupling approximation $s_\alpha(i\omega)$ vs. index of diagonality $\rho(\mathbf{D}_\alpha)$ of the damping matrix.

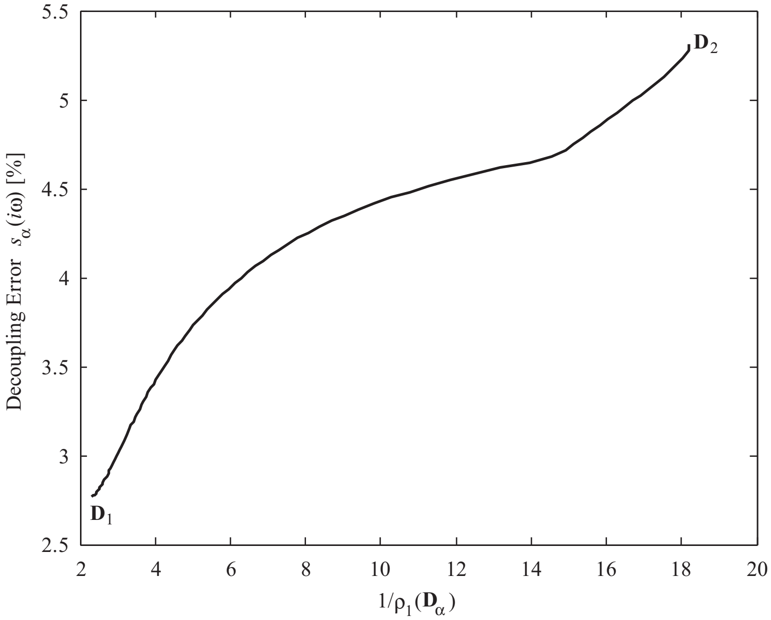

Fig. 2. Error due to the decoupling approximation $s_\alpha(i\omega)$ vs. $1/\rho_1(\mathbf{D}_\alpha)$ of the damping matrix.

monotonically from 2.76% to 5.31% as $\rho_1(\mathbf{D}_\alpha)$ decreases continuously from 0.43 to 0.055, or as $1/\rho_1(\mathbf{D}_\alpha)$ increases from 2.33 to 18.18. The error-paths in Figs. 1 and 2 are very similar. Both indicate that as the modal damping matrix becomes more diagonally dominant, errors due to the decoupling approximation increase monotonically. Thus diagonal dominance of the damping matrix is not sufficient to ensure small errors due to the decoupling approximation. There are two issues worth closer examination.

4.2. Nonlinear interpolation between damping matrices

It is certainly possible to define a nonlinear interpolation between the end states $\mathbf{D}_1$ and $\mathbf{D}_2$. Instead of the linear path given by Eq. (34), one could for example use a power law of the type

$$\mathbf{D}_\alpha = (1-\alpha^n)\mathbf{D}_1 + \alpha^n\mathbf{D}_2, \quad 0 \leqslant \alpha \leqslant 1. \tag{35}$$

Nevertheless, Eq. (35) also generates Figs. 1 and 2 because the same intermediate states $\mathbf{D}_\alpha$ are produced as α increases from 0 to 1. No new features will be observed using a nonlinear path between $\mathbf{D}_1$ and $\mathbf{D}_2$.

4.3. Variation of driving frequency

The steady-state error $s_\alpha(\mathrm{i}\omega)$ is a function of the excitation frequency. Although Figs. 1 and 2 are generated with a single driving frequency, qualitatively consistent observations can be made at any driving frequency. Suppose the driving frequency ω in Eq. (27) varies from 0 to 8 rad/s. In Fig. 3, the decoupling error $s_\alpha(\mathrm{i}\omega)$ is plotted as a function of both $\rho(\mathbf{D}_\alpha)$ and ω as a three-dimensional error surface. With any fixed ω, the error $s_\alpha(\mathrm{i}\omega)$ increases monotonically as the index of diagonality $\rho(\mathbf{D}_\alpha)$ increases continuously. Again, a graph similar to Fig. 3 is obtained if $1/\rho_1(\mathbf{D}_\alpha)$ replaces $\rho(\mathbf{D}_\alpha)$.

Example 2. In many ways, the previous example is rather conservative. It will now be demonstrated that error-paths can be constructed to increase at any specified rate while the modal damping matrix becomes more diagonally dominant. Consider a collection of four-degree-of-freedom linear systems. System 1 is the same as in Example 1, specified by Eqs. (24)–(27). Systems 3–5 are governed by

$$\ddot{\mathbf{q}} + \mathbf{D}_j\dot{\mathbf{q}} + \mathbf{\Omega}_1\mathbf{q} = \mathbf{g}(t), \quad j = 3, 4, 5, \tag{36}$$

which differ from Eq. (24) only in the off-diagonal elements of the modal damping matrices:

$$\mathbf{D}_3 = \begin{bmatrix} 1.61 & 0.0947 & -0.0140 & 0.3911 \\ 0.0947 & 1.7 & 0.3367 & -0.0125 \\ -0.0140 & 0.3367 & 1.8 & -0.0140 \\ 0.3911 & -0.0125 & -0.0140 & 1.75 \end{bmatrix}, \tag{37}$$

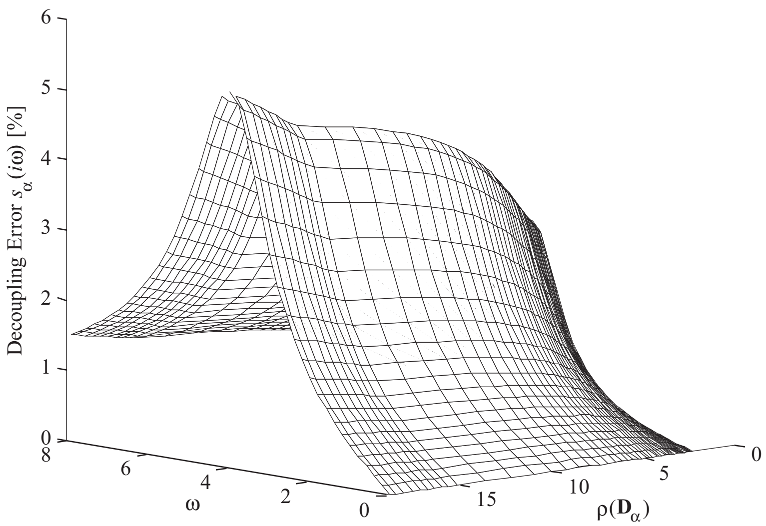

Fig. 3. Error due to the decoupling approximation $s_\alpha(\mathrm{i}\omega)$ vs. index of diagonality $\rho(\mathbf{D}_\alpha)$ and driving frequency ω.

$$\mathbf{D}_4 = \begin{bmatrix} 1.61 & 0.0762 & 0.0008 & 0.1142 \\ 0.0762 & 1.7 & 0.2090 & 0.0006 \\ 0.0008 & 0.2090 & 1.8 & 0.0388 \\ 0.1142 & 0.0006 & 0.0388 & 1.75 \end{bmatrix}, \tag{38}$$

$$\mathbf{D}_5 = \begin{bmatrix} 1.61 & -0.0008 & 0.0863 & 0.1047 \\ -0.0008 & 1.7 & 0.0380 & 0.0006 \\ 0.0863 & 0.0380 & 1.8 & 0.0863 \\ 0.1047 & 0.0006 & 0.0863 & 1.75 \end{bmatrix}. \tag{39}$$

All damping matrices $\mathbf{D}_j$ are diagonally dominant, with

$$\rho(\mathbf{D}_1) = 2.32 < \rho(\mathbf{D}_3) = 3.97 < \rho(\mathbf{D}_4) = 7.80 < \rho(\mathbf{D}_5) = 10.83, \tag{40}$$

$$\rho_1(\mathbf{D}_1) = 0.43 > \rho_1(\mathbf{D}_3) = 0.26 > \rho_1(\mathbf{D}_4) = 0.14 > \rho_1(\mathbf{D}_5) = 0.11. \tag{41}$$

Thus $\mathbf{D}_5$ is more diagonal than $\mathbf{D}_4$, $\mathbf{D}_4$ is more diagonal than $\mathbf{D}_3$, and $\mathbf{D}_3$ is more diagonal than $\mathbf{D}_1$. Intuitively, one would expect that System 1 has the largest decoupling error among the four systems. However, calculation of the steady-state errors due to the decoupling approximation yields an opposite result:

$$s_1(\mathrm{i}\omega) = 2.76\% < s_5(\mathrm{i}\omega) = 9.67\% < s_4(\mathrm{i}\omega) = 12.76\% < s_3(\mathrm{i}\omega) = 23.04\%. \tag{42}$$

In analogy to Example 1, define linear paths between $\mathbf{D}_1$ and $\mathbf{D}_j$ by

$$\mathbf{D}_{j\alpha} = (1-\alpha)\mathbf{D}_1 + \alpha\mathbf{D}_j, \quad j = 3, 4, 5, \quad 0 \leqslant \alpha \leqslant 1. \tag{43}$$

As α increases from 0 to 1, $\mathbf{D}_{j\alpha}$ is linearly interpolated between $\mathbf{D}_1$ and $\mathbf{D}_j$ with the diagonal entries of $\mathbf{D}_{j\alpha}$ remaining fixed. In Fig. 4, the decoupling error $s_{j\alpha}(\mathrm{i}\omega)$ is plotted against the index of diagonality $\rho(\mathbf{D}_{j\alpha})$ for $j = 3$, 4, 5. Three error-curves of vastly different gradients are obtained, each demonstrating that the decoupling error increases monotonically as the modal damping matrix becomes more diagonally dominant. For instance, the error-path between $(\mathbf{I}, \mathbf{D}_1, \mathbf{\Omega}_1)$ and $(\mathbf{I}, \mathbf{D}_3, \mathbf{\Omega}_1)$ indicates that the decoupling error increases

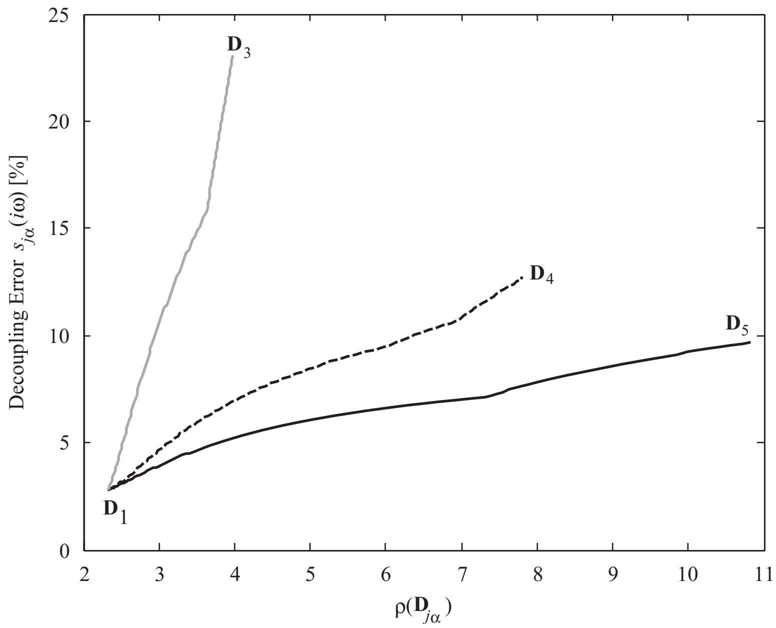

Fig. 4. Error due to the decoupling approximation $s_{j\alpha}(\mathrm{i}\omega)$ vs. $\rho(\mathbf{D}_{j\alpha})$. Error paths between $\mathbf{D}_1$ and $\mathbf{D}_3$ (▬▬), $\mathbf{D}_1$ and $\mathbf{D}_4$ (- - - ·), $\mathbf{D}_1$ and $\mathbf{D}_5$ (——) are shown.

monotonically from 2.76% to 23.04% when the index of diagonality $\rho(\mathbf{D}_{3\alpha})$ increases continuously from 2.32 to 3.97.

It is obvious from Fig. 4 that an error-path with any gradient can be constructed to originate from $(\mathbf{I}, \mathbf{D}_1, \mathbf{\Omega}_1)$. Again, the choice of an index of diagonality is immaterial and, as shown in Fig. 5, qualitatively identical results are obtained if $1/\rho_1(\mathbf{D}_{j\alpha})$ replaces $\rho(\mathbf{D}_{j\alpha})$. As previously explained, the same error-curves in Figs. 4 and 5 are generated if $\mathbf{D}_1$ and $\mathbf{D}_j$ are nonlinearly interpolated. Although Figs. 4 and 5 are generated with a single driving frequency, qualitatively consistent observations can be made at any driving frequency. In Fig. 6, the decoupling

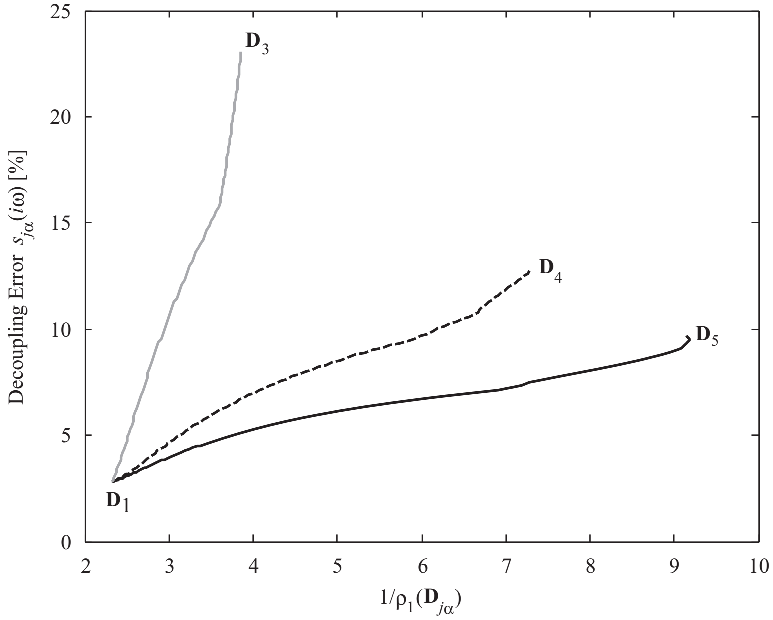

Fig. 5. Error due to the decoupling approximation $s_{j\alpha}(i\omega)$ vs. $1/\rho_1(\mathbf{D}_{j\alpha})$. Error paths between $\mathbf{D}_1$ and $\mathbf{D}_3$ (▬▬), $\mathbf{D}_1$ and $\mathbf{D}_4$ (- - -·), $\mathbf{D}_1$ and $\mathbf{D}_5$ (——) are shown.

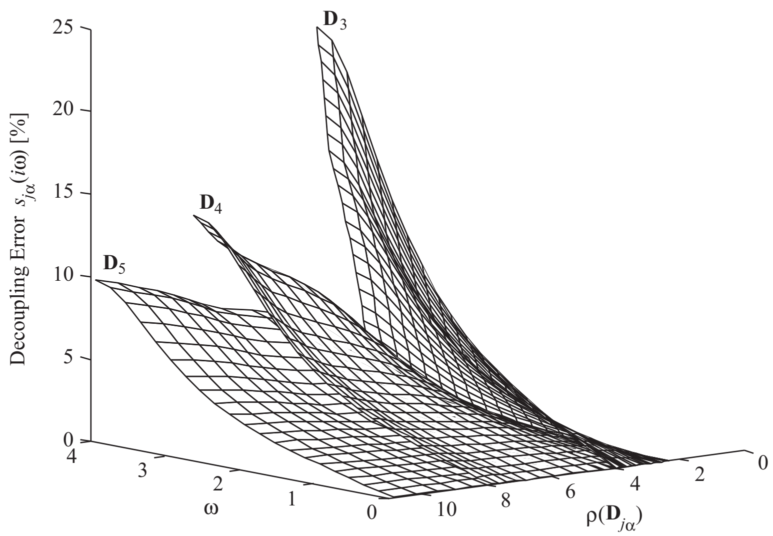

Fig. 6. Error due to the decoupling approximation $s_{j\alpha}(i\omega)$ vs. index of diagonality $\rho(\mathbf{D}_{j\alpha})$ and driving frequency ω. Error-surface between $\mathbf{D}_1$ and $\mathbf{D}_3$, $\mathbf{D}_1$ and $\mathbf{D}_4$, $\mathbf{D}_1$ and $\mathbf{D}_5$ appear, respectively, as upper, middle, and lower surfaces. To facilitate visualization, a limited range of ω is used.

error $s_{j\alpha}(\mathrm{i}\omega)$ is plotted as a function of both $\rho(\mathbf{D}_{j\alpha})$ and ω as three-dimensional error surfaces for $j = 3, 4, 5$. To facilitate visualization, the driving frequency is limited to $0 \leqslant \omega \leqslant 4$ rad/s. The three error surfaces in Fig. 6 only intersect along $\omega = 0$ and each error surface is similar in shape to the one depicted in Fig. 3. The decoupling error depends on the natural and driving frequencies. For any fixed ω, the error $s_{j\alpha}(\mathrm{i}\omega)$ increases monotonically as the index of diagonality $\rho(\mathbf{D}_{j\alpha})$ increases continuously. Again, a graph similar to Fig. 6 is obtained if $1/\rho_1(\mathbf{D}_\alpha)$ replaces $\rho(\mathbf{D}_\alpha)$.

It is clear by now that many other examples can be constructed to demonstrate that decoupling errors can increase monotonically at any specified rate as the modal damping matrix becomes more diagonally dominant with its off-diagonal elements decreasing continuously in magnitude. In fact, it is possible to construct a monotonic error-path between any two end-state systems, say $(\mathbf{I}, \mathbf{D}_1, \mathbf{\Omega}_1)$ and $(\mathbf{I}, \mathbf{D}_2, \mathbf{\Omega}_1)$, to which the decoupling approximation is applied. In the limiting case when $\mathbf{D}_2$ is diagonal, the decoupling error associated with $(\mathbf{I}, \mathbf{D}_2, \mathbf{\Omega}_1)$ is zero and only a monotonic decreasing error-path into $(\mathbf{I}, \mathbf{D}_2, \mathbf{\Omega}_1)$ can be constructed.

5. Decomposition of decoupling errors

In order to explain the rather surprising observations of the previous section, new theoretical developments will be pursued. In the frequency domain, error due to the decoupling approximation is quantified by $\|\mathbf{E}(\mathrm{i}\omega)\|$, where $\mathbf{E}(\mathrm{i}\omega)$ is given by Eq. (15). The magnitude of the kth element of $\mathbf{E}(\mathrm{i}\omega)$ represents the contribution of the kth mode to the overall decoupling error. From Eq. (15),

$$E_k(\mathrm{i}\omega) = \frac{-\mathrm{i}\omega}{\omega_k^2 - \omega^2 + \mathrm{i}\omega d_{kk}} \sum_{\substack{l=1 \\ l \neq k}}^{n} d_{kl} Q_l(\mathrm{i}\omega). \tag{44}$$

5.1. Coupling coefficients

In Eq. (44), the term

$$\varepsilon_{kl}(\mathrm{i}\omega) = \frac{-\mathrm{i}\omega}{\omega_k^2 - \omega^2 + \mathrm{i}\omega d_{kk}} d_{kl} Q_l(\mathrm{i}\omega) \tag{45}$$

arises because of coupling exerted by the lth mode on the kth mode. For this reason, $\varepsilon_{kl}(\mathrm{i}\omega)$ may be referred to as a coupling coefficient [47,48]. By definition, $\varepsilon_{kk}(\mathrm{i}\omega) = 0$. Since

$$|\varepsilon_{kl}(\mathrm{i}\omega)| = \frac{\omega}{\sqrt{(\omega_k^2 - \omega^2)^2 + \omega^2 d_{kk}^2}} |d_{kl} Q_l(\mathrm{i}\omega)|, \tag{46}$$

the amplitude of $\varepsilon_{kl}(\mathrm{i}\omega)$ is generally large if the off-diagonal elements d_{kl} of $\mathbf{D}$ is large. In terms of the complex coupling coefficients,

$$E_k(\mathrm{i}\omega) = \sum_{\substack{l=1 \\ l \neq k}}^{n} \varepsilon_{kl}(\mathrm{i}\omega). \tag{47}$$

Thus the magnitude of $E_k(\mathrm{i}\omega)$ depends on both the amplitudes and angular orientations of the coupling coefficients $\varepsilon_{kl}(\mathrm{i}\omega)$ in the complex plane. Coupling coefficients with large amplitudes may not sum to a large constituent error $E_k(\mathrm{i}\omega)$. Depending on the angular orientations of $\varepsilon_{kl}(\mathrm{i}\omega)$, the coupling coefficients may cancel out to produce a small constituent error $E_k(\mathrm{i}\omega)$. On the other hand, coupling coefficients with relatively small amplitudes can align in the complex plane to produce an unexpectedly large error.

Any assessment of errors due to the decoupling approximation based solely upon the diagonal dominance of $\mathbf{D}$ would not be accurate since such assessment does not take into account the alignment of coupling coefficients in the complex plane. This is the reason why small off-diagonal elements in $\mathbf{D}$ are not sufficient to ensure small decoupling errors.

5.2. Explanation of observations in previous examples

The systems specified as end-states in Examples 1 and 2 are summarized in Table 1. These systems differ only in the off-diagonal elements of their modal damping matrices. In Fig. 7, all coupling coefficients $\varepsilon_{kl}(\mathrm{i}\omega)$ for System 1 are plotted in the complex plane. It is also shown how these complex coupling coefficients add up to form the constituent error $E_k(\mathrm{i}\omega)$ for each of the four modes. Similar graphs are generated in Fig. 8 for System 2. Since the modal damping matrix $\mathbf{D}_2$ is significantly more diagonally dominant than $\mathbf{D}_1$, the amplitudes $|\varepsilon_{kl}(\mathrm{i}\omega)|$ of the coupling coefficients for System 2 are much smaller than the corresponding ones for System 1. However, the angular orientations of the coupling coefficients are such that they align to produce a constituent error $E_k(\mathrm{i}\omega)$ with larger magnitude in each of the four modes for System 2. This is the reason why the decoupling error associated with System 2 is larger than with System 1 even though $\mathbf{D}_2$ is significantly more diagonally dominant than $\mathbf{D}_1$.

When intermediate damping matrices $\mathbf{D}_\alpha$ are generated by Eq. (34) or (35), each $\mathbf{D}_\alpha$ is more diagonal than $\mathbf{D}_1$. However, the coupling coefficients of $(\mathbf{I}, \mathbf{D}_\alpha, \mathbf{\Omega}_1)$ align in the complex plane to produce larger constituent errors than those of $(\mathbf{I}, \mathbf{D}_1, \mathbf{\Omega}_1)$. This fully explains the counter-intuitive error-paths in Figs. 1–3 for

Table 1
Systems used as end-states in Examples 1 and 2

	System 1	System 2	System 3	System 4	System 5
System parameters	$(\mathbf{I}, \mathbf{D}_1, \mathbf{\Omega}_1)$	$(\mathbf{I}, \mathbf{D}_2, \mathbf{\Omega}_1)$	$(\mathbf{I}, \mathbf{D}_3, \mathbf{\Omega}_1)$	$(\mathbf{I}, \mathbf{D}_4, \mathbf{\Omega}_1)$	$(\mathbf{I}, \mathbf{D}_5, \mathbf{\Omega}_1)$
$\rho\ (\mathbf{D}_j)$	2.32	18.83	3.97	7.80	10.83
$\rho_1\ (\mathbf{D}_j)$	0.43	0.055	0.26	0.14	0.11
Decoupling error, s_j (iω) (%)	2.76	5.31	23.04	12.76	9.67

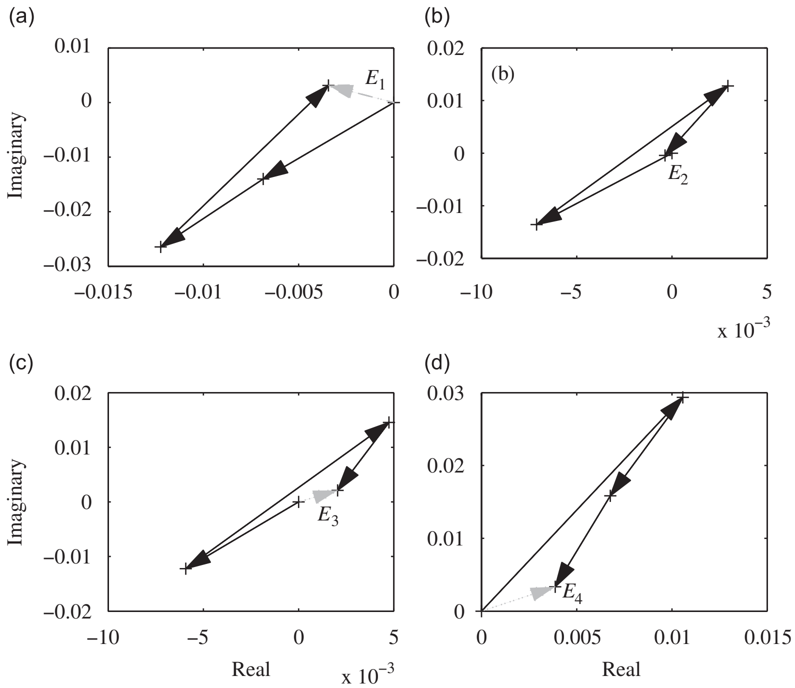

Fig. 7. Addition of coupling coefficients (——▶) for System 1 to generate the constituent error $E_k(\mathrm{i}\omega)$ (- - - -▶) in each mode: (a) Mode 1 with $|E_1| = 6.2 \times 10^{-3}$, (b) Mode 2 with $|E_2| = 4.5 \times 10^{-3}$, (c) Mode 3 with $|E_3| = 7.5 \times 10^{-3}$, and (d) Mode 4 with $|E_4| = 8.9 \times 10^{-3}$. Relatively large coupling coefficients add up in the complex plane to produce a diminished $E_k(\mathrm{i}\omega)$ in each mode.

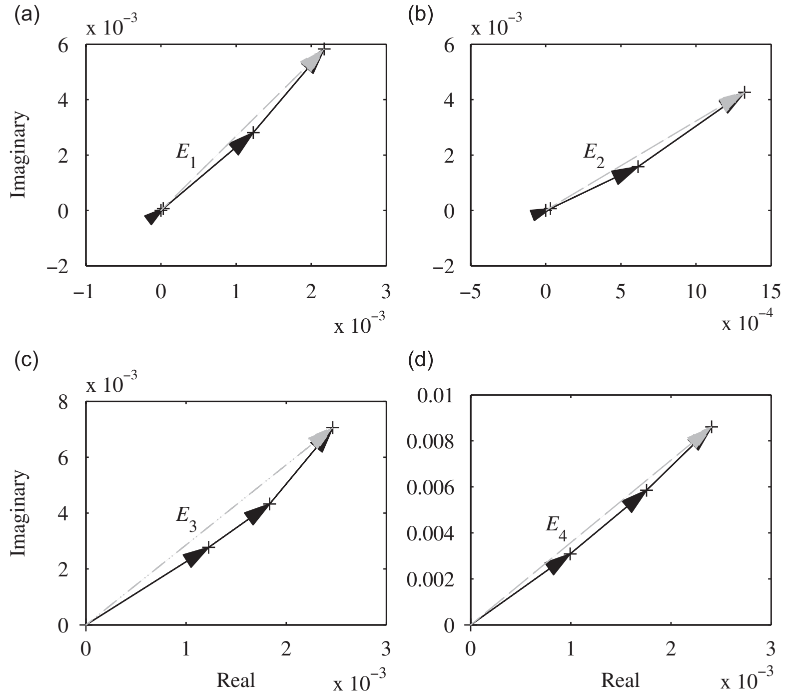

Fig. 8. Addition of coupling coefficients (⟶) for System 2 to generate the constituent error $E_k(i\omega)$ (- - - ▸) in each mode: (a) Mode 1 with $|E_1| = 4.6 \times 10^{-3}$, (b) Mode 2 with $|E_2| = 0.6 \times 10^{-3}$, (c) Mode 3 with $|E_3| = 3.0 \times 10^{-3}$, and (d) Mode 4 with $|E_4| = 5.1 \times 10^{-3}$. Relatively small coupling coefficients add up in the complex plane to produce a large $E_k(i\omega)$ in each mode.

Example 1. A similar situation exists for the systems defined in Example 2. Detailed plots of coupling coefficients for $(\mathbf{I}, \mathbf{D}_3, \mathbf{\Omega}_1)$, $(\mathbf{I}, \mathbf{D}_4, \mathbf{\Omega}_1)$, and $(\mathbf{I}, \mathbf{D}_5, \mathbf{\Omega}_1)$ in Example 2 are therefore omitted. Through an examination of the coupling coefficients in the complex plane, many unexpected behaviors in the drifts of decoupling errors can be fully explained.

6. Concluding remarks

The modal coordinates of a non-classically damped linear system are coupled by the off-diagonal elements of the modal damping matrix. A common procedure, termed the decoupling approximation, is to ignore these off-diagonal elements if they are small. The decoupling approximation, discussed by Rayleigh [19] in 1894, is a member of the class of approximate techniques seeking to decouple a linear system by modifying its coefficient matrices. It is optimal in the sense that the difference (matrix norm) between the original equation and the equation obtained by the decoupling approximation is minimized [22]. This does not mean, however, that errors due to the decoupling approximation are minimized. In this paper, errors due to the decoupling approximation have been explored both computationally and theoretically. Some observations much contrary to intuition and to widely accepted belief have been brought up and explained. Major findings of this paper are summarized in the following statements:

1. Two indices have been introduced to quantify the degree of being diagonal in damping matrices. The choice of an index of diagonality is immaterial in characterizing the drifts of errors due to the decoupling approximation.
2. Over a finite range, errors due to the decoupling approximation can increase monotonically at any specified rate while the modal damping matrix becomes more diagonally dominant with its off-diagonal elements decreasing continuously in magnitude. For instance, the error-path between $(\mathbf{I}, \mathbf{D}_1, \mathbf{\Omega}_1)$ and $(\mathbf{I}, \mathbf{D}_3, \mathbf{\Omega}_1)$ in Fig. 4 indicates that the decoupling error increases monotonically from 2.76% to 23.04% when the index of diagonality $\rho(\mathbf{D}_{3\alpha})$ increases continuously from 2.32 to 3.97.

3. Small off-diagonal elements in the modal damping matrix are not sufficient to ensure small errors due to the decoupling approximation. An error-criterion based solely upon the diagonal dominance of the modal damping matrix would not be accurate.
4. Any assessment of errors due to the decoupling approximation should utilize both the amplitudes and angular orientations of the complex coupling coefficients. A monotonic error-path can be constructed between any two end-state systems, say $(\mathbf{I}, \mathbf{D}_1, \mathbf{\Omega}_1)$ and $(\mathbf{I}, \mathbf{D}_2, \mathbf{\Omega}_1)$, to which the decoupling approximation is applied.

Although a limited set of data is presented herein, extensive calculations have been performed by the authors, and all numerical simulations have yielded qualitatively identical results. Drifts in errors due to the decoupling approximation are not as intuitive and simplistic as are usually thought. Through research into characterization of decoupling errors, it is hoped that the decoupling approximation can be used by practicing engineers with increased confidence and discretion in the years to come.

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